Geometric Folding Algoritihms: Linkages, Origami, Polyhedra

Updates to Chapter 22.5: Special Classes of Edge-Unfoldable Polyhedra

Erik D. Demaine Joseph O'Rourke

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1 Introduction

Updates to Chapter 22, Edge-Unfolding of Polyhedra [DO07], specifically to Section 22.5, Special Classes of Edge-Unfoldable Polyhedra.

We use the term *net* to mean a non-overlapping unfolding of a polyhedron $\mathcal{P}$ to a polygon P. In particular, P avoids *strong overlap*, an overlap of two interior points. But *weak overlap*—boundary-to-boundary touching—is permitted (and thus P could be cut out of paper). So a net is a *weakly simple polygon* [CEX14]. This distinction between strong and weak overlap was not emphasized in *Geometric Folding Algoritihms: Linkages, Origami, Polyhedra* (GFALOP).

Our focus throughout Chapter 22 is edge-unfolding to a net.

2 In GFALOP

The following infinite classes of polyhedra are discussed in Section 22.5

(1) Pyramids[1]

(2) Prisms

(3) Prismoids

(4) Domes

(5) Convex generalized deltahedra[2]

[1] Not explicitly discussed but a subset of domes.

[2] Originally called "higher-order" deltahedra. See [Bez07].

(6) Gridded orthotubes [BDD+98]. (See discussion in Section 3.2.)

Note that prismatoids (for which edge-unfolding remains open) includes prismoids which includes prisms.

3 Post GFALOP

The following additional classes of polyhedra have been proved to edge-unfold to nets.

3.1 Convex Polyhedra

(7) Every convex polyhedron can be stretched via an affine transformation so that it has an edge-unfolding to a net [Gho14],

(8) Nearly flat acutely triangulated convex cap, even if closed to a polyhedron by adding the convex polygonal base under the cap [O'R18] [O'R17].

(9) Prismatoids: Tall or Rectangular Base [BDM21].

(10) Polyhedra with a Hamiltonian simple closed quasigeodesic [O'R22]:

(11) Polar zonohedra [O'R23].

(12) Nested prismatoids [Rad24].

(13) Tapered polyhedra; each is "skeletal" [OV24].

3.2 Gridded Orthogonal Polyhedra

Gridded orthogonal polyhedra allow cuts along the edges produced by slicing by coordinate planes through every vertex. Here we focus on unrefined grid-edge faces, i.e., a 1×1 grid.

(14) Orthogonal Terrains [O'R07].

(15) Rectangle-faced orthostacks [CST12].

(16) Orthotubes [DK22].

(17) Grid-edge unfolding orthostacks with rectangular slabs [Per24].

3.3 Polycubes

A *polycube* is an orthogonal polyhedron built by face-to-face gluings of (unit) cubes. These are a special case of *gridded orthogonal polyhedra*, listed above.. It is natural to extend the notion of edge-unfolding to allow cuts along cube edges, which may not be edges of the corresponding orthogonal polyhedron. The main driver here is Open Problem 22.4, which remains open: Is there any genus-0 polycube that cannot be edge-unfolded (under the extended notion of edge-unfolding)? A negative result for special "edge zipper unfoldings" is [DDEO22].

(18) Well-separated orthotrees [DFMO05].

(19) One layer polycubes with sparse cubic holes [LPW14].

(20) One layer polycubes with general cubic holes [Min22].

(21) Polycubes with orthogonally convex layers [DM24].

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